

# Absence of Phase Transitions in a Class of Integer Spin Systems

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**Abstract** We exhibit a class of integer spin systems whose free energy can be written in term of an absolutely convergent series at any temperature. This class includes spin systems on  $\mathbb{Z}^d$  interacting through infinite range pair potential polynomially decaying at large distances  $r$  at a rate  $1/r^{d+\varepsilon}$  with  $\varepsilon > 0$ . It also contains the Blume-Emery-Griffiths model in the disordered phase at large values of the crystal field.

**Keywords** Integer spin systems · Cluster expansion

## 1 The Class of Spin Systems: Definitions and Results

Let  $\mathbb{V}$  be a countable set. We define an integer spin system on  $\mathbb{V}$  by supposing that in each site  $x \in \mathbb{V}$  there is a random variable  $\sigma_x$  (the spin at  $x$ ) taking values in the set  $\{0, \pm 1, \pm 2, \dots, \pm N\}$  where  $N$  is an integer. For  $U \subset \mathbb{V}$ , a spin configuration  $\sigma_U$  in  $U$  is a function  $\sigma_U : U \rightarrow \{0, \pm 1, \pm 2, \dots, \pm N\} : x \mapsto \sigma_x$ . We denote by  $\Sigma_U$  the set of all spin configurations in  $U$ . Given a spin configuration  $\sigma_\Lambda$  in the finite “volume”  $\Lambda \subset \mathbb{V}$ , the Hamiltonian of the system at volume  $\Lambda$  (with free boundary conditions) is defined as

$$H_{\Lambda(\omega)} = \sum_{\{x,y\} \subset \Lambda} V(x, y, \sigma_x, \sigma_y) + D \sum_{x \in \Lambda} \sigma_x^2 \quad (1.1)$$

Typically, the set  $\mathbb{V}$  is the unit cubic lattice  $\mathbb{Z}^d$ , but theorem 1 below holds for any countable set  $\mathbb{V}$  regardless of its structure. The assumptions on the pair potential  $V(x, y, \sigma_x, \sigma_y)$  in the Hamiltonian (1.1) are the following.

**A.** For any pairs  $x, y$  such that  $\sigma_x \sigma_y = 0$  we have

$$V(x, y, \sigma_x, \sigma_y) = 0 \quad (1.2)$$

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**B.** There exists a positive function  $J(x, y)$  and a positive number  $J$  such that

$$|V(x, y, \sigma_x, \sigma_y)| \leq J(x, y), \quad \forall (\{x, y\}, \sigma_x, \sigma_y) \tag{1.3}$$

and

$$\sup_{x \in \mathbb{V}} \sum_{\substack{y \in \mathbb{V} \\ y \neq x}} J(x, y) = 2J \tag{1.4}$$

Note that Assumption (1.4) includes infinite range interactions polynomially decaying in a summable way. Note also that, when  $D > J$ , the lowest-energy state (i.e. the spin configuration  $\sigma_\Lambda^0 \in \Sigma_\Lambda$  for which  $H_\Lambda(\sigma_\Lambda)$  attains its minimum) is the spin configuration  $\sigma_\Lambda^0 = 0$ , i.e. all sites have spin equal to zero. The region  $D > J$  is usually called the disordered phase and the parameter  $D$  is called the crystal field.

The partition function of the system in the volume  $\Lambda$ , at inverse temperature  $\beta$  is given by

$$Z_\Lambda(\beta) = \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{-\beta H(\sigma_\Lambda)} \tag{1.5}$$

and the free energy at finite volume  $\Lambda$  is given by

$$f_\Lambda(\beta) = \frac{1}{|\Lambda|} \ln Z_\Lambda(\beta) \tag{1.6}$$

where here and elsewhere in the paper  $|\Lambda|$  is the number of elements of the finite set  $\Lambda$ .

Our results on the system above can be summarized by the following theorem.

**Theorem 1** *Consider the spin system with Hamiltonian (1.1). Under the Assumptions (1.2)–(1.4), there exist positive numbers  $D_c, \beta_1$  and  $\beta_2$  with  $D_c > J$  and  $\beta_1 < \beta_2$  such that the free energy  $f_\Lambda(\beta)$  defined in (1.6) can be written in terms of an absolutely convergent series uniformly bounded in  $\Lambda$  either if  $\beta \in [0, \beta_1]$  or  $\beta \in [\beta_2, \infty)$  and  $D > J$ . Moreover if  $D \geq D_c$  then  $f_\Lambda(\beta)$  converges absolutely for all  $\beta \in [0, \infty)$ .*

We remind the reader that the standard polymer expansion has been shown to be absolutely convergent at sufficiently high temperatures or low density for wide class of spin systems, either using Kirkwood-Salzburg equations [7, 8] or by bounding directly the coefficients of the expansion [17]. Moreover it has been shown in [9] that analyticity of a similar class of spin systems at high temperatures or low densities can be inferred from the Dobrushin uniqueness theorem.

On the other hand it is well known that for a wide class of spin systems a unique zero-temperature ground state implies a unique Gibbs state at sufficiently low temperatures and hence absence of phase transitions (see e.g. [12] and references therein). Analyticity at low-temperature via Pirogov-Sinai theory has been established in [19] for spin systems interacting via finite-range potentials and has been extended in [13] to infinite-range potentials.

Here we present a class of spin systems, depending on a positive parameter  $D$ , whose free energy is analytic simultaneously at high and low temperatures if  $D > J$  or at any temperature, if  $D > D_c$ . It is worth remarking that our class of spin systems also includes the Blume-Emery-Griffiths (BEG) model [4] on  $\mathbb{Z}^d$ . To recover the BEG model just choose  $\mathbb{V} = \mathbb{Z}^d, N = 1$  and

$$V(x, y, \sigma_x, \sigma_y) = -V_{\{x,y\}}\sigma_x\sigma_y + K_{\{x,y\}}\sigma_x^2\sigma_y^2$$

with  $V_{\{x,y\}} = V > 0$ ,  $K_{\{x,y\}} = K \in \mathbb{R}$  if  $x, y$  are nearest neighbors and  $V_{\{x,y\}} = K_{\{x,y\}} = 0$  otherwise.

Specifically on the BEG model, there are some rigorous results [5, 11] concerning the uniqueness of the equilibrium state in the regime where the ferromagnetic interaction  $V$  is sufficiently strong. Our results imply that no phase transition can occur in the BEG model when the crystal field is sufficiently large.

## 2 Proof of Theorem 1

We will prove the theorem by performing a high temperature polymer expansion on the partition function of the system. We will see below that such an expansion, when applied to a spin system described by the Hamiltonian (1.1), converges also in the region of low temperatures (large  $\beta$ ) as soon as  $D > J$  and the gap between the analytic high temperature phase and the analytic low-temperature phase depends on the strength of the crystal field  $D$ . When  $D$  is sufficiently large, this gap disappears.

We begin, by rewriting the partition function (1.5) of the system in the volume  $\Lambda \subset \mathbb{V}$  as follow.

$$Z_\Lambda(\beta) = \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{-\beta H(\sigma_\Lambda)} = \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{-\beta D \sum_{x \in \Lambda} \sigma_x^2} e^{-\beta \sum_{\{x,y\} \subset \Lambda} V(x,y,\sigma_x,\sigma_y)} \tag{2.1}$$

We then expand the exponential in l.h.s. of 2.1) as follows.

$$\begin{aligned} e^{-\beta \sum_{\{x,y\} \subset \Lambda} V(x,y,\sigma_x,\sigma_y)} &= \prod_{\{x,y\} \subset \Lambda} [e^{-\beta \sum_{\{x,y\} \subset \Lambda} V(x,y,\sigma_x,\sigma_y)} - 1 + 1] \\ &= \sum_{s=1}^{|\Lambda|} \sum_{\{R_1, \dots, R_s\} \in \pi_s(\Lambda)} \rho(R_1, \sigma_{R_1}) \cdots \rho(R_s, \sigma_{R_s}) \end{aligned}$$

Here  $\pi_s(\Lambda)$  denotes the set of all partitions of  $\Lambda$  in  $s$  non empty subsets, and

$$\rho(R, \sigma_R) = \begin{cases} 1 & \text{if } |R| = 1 \\ \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] & \text{if } |R| \geq 2 \end{cases}$$

with  $G_R$  being the set of all connected graphs with vertex set  $R$  ( $E_g$  denotes the edge set of a graph  $g \in G_R$ ). Thus (2.1) becomes,

$$\begin{aligned} Z_\Lambda(\beta) &= \sum_{\sigma_\Lambda \in \Sigma_\Lambda} e^{-\beta D \sum_{x \in \Lambda} \sigma_x^2} \sum_{s=1}^{|\Lambda|} \sum_{R_1, \dots, R_s \in \pi_s(\Lambda)} \rho(R_1, \sigma_{R_1}) \cdots \rho(R_s, \sigma_{R_s}) \\ &= \sum_{s=1}^{|\Lambda|} \sum_{R_1, \dots, R_s \in \pi_s(\Lambda)} \tilde{\rho}(R_1) \cdots \tilde{\rho}(R_s) \end{aligned}$$

where

$$\tilde{\rho}(R) = \sum_{\sigma_R \in \Sigma_R} \rho(R) e^{-\beta D \sum_{x \in R} \sigma_x^2}$$

Now, due to assumption **A** formula (1.2), we have that  $\prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] = 0$  for any  $g \in G_R$  whenever  $\sigma_x = 0$  for some  $x \in R$ . Hence a straightforward computation gives

$$\tilde{\rho}(R) = \begin{cases} 1 + 2 \sum_{k=1}^N e^{-\beta Dk^2} & \text{if } |R| = 1 \\ \sum_{\sigma_R \in \tilde{\Sigma}_R} e^{-\beta D \sum_{x \in R} \sigma_x^2} \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] & \text{if } |R| \geq 2 \end{cases}$$

where  $\tilde{\Sigma}_R = \{\sigma_R \in \Sigma_R : \sigma_x \neq 0, \forall x \in R\}$ .

Define now, for any finite  $R \subset \mathbb{V}$  such that  $|R| \geq 2$

$$\zeta(R) = \left[ \frac{1}{1 + 2 \sum_{k=1}^N e^{-\beta Dk^2}} \right]^{|R|} \sum_{\sigma_R \in \tilde{\Sigma}_R} e^{-\beta D \sum_{x \in R} \sigma_x^2} \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] \tag{2.2}$$

Then it is not difficult to check that

$$Z_\Lambda(\beta) = \left( 1 + 2 \sum_{k=1}^N e^{-\beta Dk^2} \right)^{|\Lambda|} \Xi_\Lambda(\beta) \tag{2.3}$$

where  $\Xi_\Lambda(\beta)$  is given by

$$\Xi_\Lambda(\beta) = 1 + \sum_{s \geq 1} \frac{1}{n!} \sum_{\substack{R_1, \dots, R_n \\ |R_i| \geq 2, R_i \cap R_j = \emptyset}} \zeta(R_1) \cdots \zeta(R_n) \tag{2.4}$$

Hence the free energy of the system is given by

$$f_\Lambda(\beta) = \log \left( 1 + 2 \sum_{k=1}^N e^{-\beta Dk^2} \right) + P_\Lambda(\beta)$$

where

$$P_\Lambda(\beta) = \frac{1}{|\Lambda|} \log \Xi_\Lambda(\beta) \tag{2.5}$$

The function  $\log(1 + 2 \sum_{k=1}^N e^{-\beta Dk^2})$  is analytic for all  $\beta \geq 0$  so to check the analyticity of the free energy it is sufficient to check the analyticity of the function  $P_\Lambda(\beta)$  defined in (2.5).

Now we just note that  $\Xi_\Lambda(\beta)$  is the partition function of a hard-core polymer gas in which the polymers  $R$  are finite subsets of  $\mathbb{V}$  with cardinality greater than 1, activity  $\zeta(R)$  and with the incompatibility relation being the non-empty intersection. By Fernandez-Procacci criterion [6] (see also Sect. 3 in [10]), the pressure  $P_\Lambda(\beta)$  of this hard-core polymer gas is written in terms of an absolutely convergent series bounded uniformly in  $\Lambda$  if the activities (2.2) satisfy

$$\inf_{a > 0} (e^a - 1)^{-1} \sum_{n \geq 2} e^{an} \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\zeta(R)| \leq 1 \tag{2.6}$$

Now, since  $e^{-\beta D \sum_{x \in R} \sigma_x^2} \leq e^{-\beta D|R|}$  for any  $\sigma_R \in \tilde{\Sigma}_R$  and setting

$$\tilde{\lambda}_\beta = \frac{e^{-\beta D}}{1 + 2 \sum_{k=1}^N e^{-\beta Dk^2}} \tag{2.7}$$

we have

$$\sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\zeta(R)| \leq \tilde{\lambda}_\beta^n \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} \sum_{\sigma_R \in \tilde{\Sigma}_R} \left| \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] \right| \tag{2.8}$$

We now need to bound the factor

$$\left| \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] \right| \tag{2.9}$$

in the r.h.s. of the equation above. Usually factors like (2.9) are bounded using sophisticated tools in cluster expansion such as the so called Battle-Brydges-Federbush tree inequality (see e.g [2, 16]) or the Brydges-Kennedy-Abdesselam-Rivasseau formula (see e.g [1, 3]). However, for the case considered here (bounded spins in a discrete set), it is possible to use a much simpler result obtained recently in [10] whose very simple proof is based on the Penrose identity [14] (see also [15, 18], and [6]). Such a result, stated in [10] as Proposition 4.3, in the present notations can be rephrased as follows.

**Proposition 2** (Jackson-Procacci-Sokal) *Let  $P_2(\mathbb{V})$  be the set of unordered pairs of distinct elements of  $\mathbb{V}$  and let  $v : P_2(\mathbb{V}) \rightarrow \mathbb{C} : \{x, y\} \mapsto v_{\{x,y\}}$  be a map with the following property: there exists a (nonnegative) constant  $B$  such that*

$$\sup_{x \in \mathbb{V}} \sum_{\substack{y \in \mathbb{V} \\ y \neq x}} |v_{\{x,y\}}| \leq 2B \tag{2.10}$$

Then, for any finite  $R \subset S$  with  $|R| \geq 2$  the following inequality holds:

$$\left| \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} (e^{-v_{\{x,y\}}} - 1) \right| \leq e^{B|R|} \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (1 - e^{-|v_{\{x,y\}}|})$$

where  $T_R$  is the set of all trees with vertex set  $R$  and  $E_\tau$  denotes the edge set of a tree  $\tau \in T_R$ .

*Proof* Choose a root  $x_0$  in  $R$  so that any tree  $\tau \in T_R$  is regarded as rooted in  $x_0$ . Choose an order in  $R$ , i.e. to any  $x$  in  $R$  associate in a one-to-one way a number in  $\{1, 2, \dots, |R|\}$  (the label of  $x$ ). Then for any tree  $\tau \in T_R$  with edge set  $E_\tau$  define a graph  $p(\tau)$  in  $G_R$  with edge set  $E_{p(\tau)} \supset E_\tau$  obtained by adding to  $E_\tau$  all unordered pairs in  $R$  connecting each vertex  $x \neq x_0$  of  $\tau$  to all its siblings in  $\tau$  (vertices of the same generation) and to all its ‘‘uncles’’ in  $\tau$  (vertices of the previous generation) with label greater than the label of the parent of  $x$ . Then the Penrose identity is

$$\sum_{g \in G_R} \prod_{\{x,y\} \in E_g} (e^{-v_{\{x,y\}}} - 1) = \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (e^{-v_{\{x,y\}}} - 1) e^{-\sum_{\{z,z'\} \in E_{p(\tau)} \setminus E_\tau} v_{\{z,z'\}}}$$

Hence

$$\left| \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} (e^{-v_{\{x,y\}}} - 1) \right| \leq \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (e^{|v_{\{x,y\}}|} - 1) e^{\sum_{\{z,z'\} \in E_{p(\tau)} \setminus E_\tau} |v_{\{z,z'\}}|}$$

$$\begin{aligned}
 &= \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (1 - e^{-|v_{\{x,y\}}|}) e^{\sum_{\{z,z'\} \in E_p(\tau)} |v_{\{z,z'\}}|} \\
 &\leq \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (1 - e^{-|v_{\{x,y\}}|}) e^{\sum_{\{z,z'\} \subset R} |v_{\{z,z'\}}|} \\
 &\leq \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (1 - e^{-|v_{\{x,y\}}|}) e^{\frac{1}{2}|R| \sup_{z \in R} \sum_{\substack{z' \in \mathbb{V} \\ z' \neq z}} |v_{\{z,z'\}}|} \leq e^{B|R|} \sum_{\tau \in T_R} \prod_{\{x,y\} \in E_\tau} (1 - e^{-|v_{\{x,y\}}|}) \quad \square
 \end{aligned}$$

Now, for  $v_{\{x,y\}} = \beta V(x, y, \sigma_x, \sigma_y)$ , by Assumption **B**, formulas (1.4) and (1.3), the pair potential  $V(x, y, \sigma_x, \sigma_y)$  satisfies (2.10) with  $B = J$ . Therefore, Proposition 2 leads to the bound

$$\left| \sum_{g \in G_R} \prod_{\{x,y\} \in E_g} [e^{-\beta V(x,y,\sigma_x,\sigma_y)} - 1] \right| \leq e^{\beta J|R|} \sum_{t \in T_R} \prod_{\{x,y\} \in E_t} (1 - e^{-\beta J(x,y)}). \tag{2.11}$$

Hence, pugging (2.11) in (2.8) and using also that  $\sum_{\sigma_R \in \tilde{\Sigma}_R} 1 = (2N)^n$ , we get

$$\begin{aligned}
 \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\zeta(R)| &\leq [2N \tilde{\lambda}_\beta e^{\beta J}]^n \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} \sum_{t \in T_R} \prod_{\{x,y\} \in E_t} (1 - e^{-\beta J(x,y)}) \\
 &= \frac{[2N \tilde{\lambda}_\beta e^{\beta J}]^n}{(n-1)!} \sup_{x \in \mathbb{V}} \sum_{t \in T_n} \sum_{\substack{(x_1, \dots, x_n) \in \mathbb{V}^n \\ x_1 = x, x_i \neq x_j}} \prod_{\{i,j\} \in E_t} (1 - e^{-\beta J(x_i, x_j)}) \tag{2.12}
 \end{aligned}$$

where  $T_n$  denotes the set of trees with vertex set  $\{1, 2, \dots, n\}$ . It is now easy to check that

$$\sum_{\substack{(x_1, \dots, x_n) \in \mathbb{V}^n \\ x_1 = x, x_i \neq x_j}} \prod_{\{i,j\} \in E_t} (1 - e^{-\beta J(x_i, x_j)}) \leq [h(\beta, J)]^{n-1}, \quad \forall t \in T_n \tag{2.13}$$

with

$$h(\beta, J) = \sup_{\substack{x \in \mathbb{V} \\ y \in \mathbb{V} \\ y \neq x}} (1 - e^{-\beta J(x,y)}) \tag{2.14}$$

Note that, by (1.4), we have

$$h(\beta, J) = \sup_{x \in \mathbb{V}} \sum_{\substack{y \in \mathbb{V} \\ y \neq x}} \beta J(x, y) \int_0^1 e^{-\beta J(x,y)t} \leq \sup_{\substack{x \in \mathbb{V} \\ y \in \mathbb{V} \\ y \neq x}} \beta J(x, y) \leq 2\beta J \tag{2.15}$$

Using finally Cayley formula (i.e.  $\sum_{t \in T_n} 1 = n^{n-2}$ ) for tree counting, we obtain

$$\sup_{x \in \mathbb{V}} \sum_{\substack{R: x \in R \\ |R|=n}} |\zeta(R)| \leq \frac{n^{n-2}}{(n-1)!} [h(\beta, J)]^{n-1} [2N \tilde{\lambda}_\beta e^{\beta J}]^n$$

Hence the convergence criterion (2.6) is

$$\inf_{a>0} (e^a - 1)^{-1} \sum_{n=2}^{+\infty} e^{an} [2Nh(\beta, J) \tilde{\lambda}_\beta e^{\beta J}]^{n-1} \frac{n^{n-1}}{n!} \leq \frac{1}{2N \tilde{\lambda}_\beta e^{\beta J}} \tag{2.16}$$

which, by Lemma 6.1 in [10] is satisfied when

$$\frac{1}{2N h(\beta, J) \tilde{\lambda}_\beta e^{\beta J}} \geq 8N \tilde{\lambda}_\beta e^{\beta J} + 3$$

i.e. when

$$e^{(D-J)\beta} F(\beta) \geq h(\beta, J) \tag{2.17}$$

with

$$F(\beta) = \frac{1}{2} \frac{(1 + 2 \sum_{k=1}^N e^{-\beta D k^2})^2}{[8N^2 e^{-(D-J)\beta} + 3N(1 + 2 \sum_{k=1}^N e^{-\beta D k^2})]} \tag{2.18}$$

Now it is easy to see that there always exist two positive numbers  $\beta_1$  and  $\beta_2$  (with  $\beta_1 < \beta_2$ ) depending on  $D, J$  and  $N$ , such that the inequality (2.17) is in general satisfied either when  $\beta \leq \beta_1$ , or when  $\beta \geq \beta_2$  and  $D > J$ .

Indeed,  $F(0) = \frac{1}{4} \frac{(1+2N)^2}{3N+14N^2}$ , so that for  $\beta$  sufficiently small, say  $\beta \leq \beta_1$ , the inequality (2.17) is always satisfied, because the l.h.s. tends to  $F(0) > 0$  for  $\beta \rightarrow 0$  regardless of the value of  $D$ , while in the r.h.s.  $\lim_{\beta \rightarrow 0} h(\beta, J) = 0$ , by (2.15).

On the other hand,  $F(\beta)$  tends to  $\frac{1}{12N}$  as  $\beta \rightarrow \infty$ . So, as soon as  $D > J$ , the inequality (2.17) is surely satisfied for  $\beta$  sufficiently large, say  $\beta \geq \beta_2$ , when the exponential on the l.h.s. of (2.17) beats the function  $h(\beta, J)$ . which, by (2.15), is always below the linear function  $2\beta J$ .

Finally, let us show that there is a critical value of  $D_c$  (depending on  $J$ ) such that, whenever  $D \geq D_c$ , the inequality (2.17) is always satisfied for all  $\beta \in [0, \infty)$ . Indeed, an upper bound for  $D_c$  can be easily found if we are not looking for optimal bounds. As a matter of fact it is easy to see that

$$F(\beta) \geq \frac{1}{6N + 16N^2}, \quad \text{for all } \beta \geq 0 \tag{2.19}$$

Moreover using the bound (2.15) the inequality (2.17) is surely satisfied if

$$e^{(D-J)\beta} \geq (12N + 32N^2)\beta J \tag{2.20}$$

which is always satisfied for any  $\beta \geq 0$  as soon as  $D \geq (1 + \frac{12N+32N^2}{e})J$ . So we get for the critical value  $D_c$  of the crystal field the upper bound

$$D_c \leq \left(1 + \frac{12N + 32N^2}{e}\right)J \tag{2.21}$$

Of course, due to the rough bounds (2.15) and (2.19), this estimate is far from being optimal. In particular, the replacement in (2.17) of the function  $h(\beta, J)$  with its upper bound (2.15) seems to be a quite crude estimate, especially in the large  $\beta$  regime. To find with accuracy the regions of validity of the inequality (2.17) so as to improve the upper bound for  $D_c$ , one should be able compute explicitly function  $h(\beta, J)$  which depends sensibly on the behavior of  $J(x, y)$  and this would go beyond the scope of the present note.

We conclude by noting that improvements on the upper bound (2.21) might also be obtained by trying to enlarge the analyticity region in the disordered phase (i.e. when  $D > J$ ) via a genuine low-temperature contour expansion for the system, i.e an expansion around the ground state  $\sigma = 0$ .

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